

Refactorizing NRQCD short-distance coefficients in exclusive quarkonium production

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Abstract

In a typical exclusive quarkonium production process, when the center-of-mass energy, \sqrt{s} , is much greater than the heavy quark mass m , large kinematic logarithms of s/m^2 will unavoidably arise at each order of perturbative expansion in the short-distance coefficients of the nonrelativistic QCD (NRQCD) factorization formalism, which may potentially harm the perturbative expansion. This symptom reflects that the hard regime in NRQCD factorization is too coarse and should be further factorized. We suggest that this regime can be further separated into “hard” and “collinear” degrees of freedom, so that the familiar light-cone approach can be employed to reproduce the NRQCD matching coefficients at the zeroth order of m^2/s and order by order in α_s . Taking two simple processes, exclusive $\eta_b + \gamma$ production in e^+e^- annihilation and Higgs boson radiative decay into Υ , as examples, we illustrate how the leading logarithms of s/m^2 in the NRQCD matching coefficients are identified and summed to all orders in α_s with the aid of Brodsky-Lepage evolution equation.

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Introduction. One of the classical applications of perturbative Quantum Chromodynamics (QCD) is the successful description of many exclusive processes with large momentum transfer using collinear factorization, that allows one to express the scattering amplitude as the convolution of perturbatively calculable short-distance parts and the nonperturbative but universal light-cone distribution amplitudes (LCDA) [1, 2]. In describing hard exclusive processes involving heavy quarkonium, *i.e.*, a nonrelativistic bound state made of a heavy quark and a heavy antiquark, however, there also exists another widely-accepted theoretical framework, the nonrelativistic QCD (NRQCD) factorization formalism [3]. In this approach, the amplitude can also be put in a factorized form, that is, an infinite sum of products of short-distance coefficients and nonperturbative, albeit universal, NRQCD matrix elements.

In recent years, considerable amount of efforts have been spent to understand exclusive charmonium production mechanisms from both NRQCD and light-cone perspectives. This endeavor is perhaps largely propelled by a somewhat unexpected finding, that the lowest-order NRQCD calculation of double charmonium production rate for the process $e^+e^- \rightarrow J/\psi\eta_c$ [4] fell short by about one order-of-magnitude of the BELLE measurement [5]. The validity of applying NRQCD to charmonia was soon questioned by some authors, who advocated that if charmonium is treated as light meson, the light-cone approach instead could satisfactorily accommodate the BELLE data [6, 7, 8]. However, a careful reexamination [9] suspected that these optimistic assertions are premature and it was argued that a “consistent” light-cone analysis would in fact yield a result not much different from NRQCD, so that the situation has not truly improved.

In reviewing this episode, one may get the impression that the light-cone and NRQCD approaches are two drastically different, and, competing, theoretical frameworks. Indeed, these two approaches are rooted in two different types of operator-product-expansions (OPE), where the former is intimately linked to the light-cone (twist) expansion, and the latter is more closely related to a local (large mass) expansion.

The main motif of this work is to show that, these two approaches, both bearing solid theoretical grounds, need not to be regarded solely as rivals. As a matter of fact, they can benefit each other ¹. In particular, it turns out that the concepts and techniques developed

¹ We note that there already exist some attempts along this line, but motivated by somewhat different considerations. In particular, there is a recent work trying to bridge the leading-twist LCDA of S -wave charmonia with the local NRQCD matrix elements [10] (see also [11, 12]).

in the light-cone approach can be fruitfully exploited to improve the NRQCD calculations. For example, we will argue that, for some class of exclusive single-quarkonium production processes, the NRQCD short-distance coefficients can be reproduced order by order in α_s , in the language of collinear factorization. Moreover, the light-cone approach can easily resum large kinematic logarithms appearing in the NRQCD short-distance coefficients.

Refactorization of NRQCD matching coefficients. NRQCD is an effective field theory that is constructed to portray the nonrelativistic character of heavy quarkonium. The slow relative motion between quark and antiquark in a quarkonium, v , naturally constitutes the expansion parameter of the theory. To produce a quarkonium, the heavy quark pair must be created within a distance shorter than $1/m$ and have a small relative velocity, to warrant a significant probability to form a quarkonium. The first condition guarantees that the parton process can be calculated perturbatively owing to asymptotic freedom. The latter requirement implies that the full amplitude can be expanded in power of v , and NRQCD endows a well-defined meaning for this Taylor expansion procedure.

For definiteness of our discussion, we will confine ourselves to the *bottomonium* throughout this work, whenever a quarkonium is referred to. Of course, all the results can be trivially copied to the charmonium case. Furthermore, for simplicity, in this work we will concentrate on the single quarkonium production process, and not touch upon the double-quarkonium production for its additional technical complication. For a typical hard process that creates a bottomonium, generically denoted by H , NRQCD factorization demands that the amplitude can be expressed as

$$\mathcal{M}[H] \sim \sum_n \mathcal{C}_n(Q, m) \langle H | \mathcal{O}_n | 0 \rangle, \quad (1)$$

where \mathcal{O}_n stands for an appropriate local color-singlet NRQCD operator, m is the b quark mass, Q refers to a typical kinematic scale such as the center-of-mass energy, presumably much greater than m . The important feature is that each \mathcal{O}_n has a definite power counting in v , so that the expansion in (1) becomes practically maneuverable.

NRQCD factorization manifestly separates the effects of *hard* degree of freedom ($p^\mu \gtrsim m$) from the remaining lower-energy ones². In Eq. (1), the matching coefficients \mathcal{C}_n encode the

² There are totally three kinds of such, *soft* ($p^\mu \sim mv$), *potential* ($p^0 \sim mv^2$, $\mathbf{p} \sim mv$) and *ultrasoft* ($p^\mu \sim mv^2$) [13]. All these scalings are specified in the quarkonium rest frame.

effects of hard quantum fluctuation, which are perturbatively calculable and infrared finite, whereas the local NRQCD matrix elements characterize the long-distance, and primarily nonperturbative, aspects of H .

The short-distance coefficient \mathcal{C}_n depends on two energy scales, Q and m . When these two scales are widely separated, it can be naturally organized by Taylor series in m^2/Q^2 , practically only the leading power term needs to be retained³. It is always possible, by factoring out an appropriate kinematic factor proportional to some powers of Q , to render \mathcal{C}_n dimensionless, and such that it only depends on m^2 and Q^2 through their dimensionless ratio and it scales with m^2/Q^2 at most logarithmically.

It is conceivable that at each order in loop expansion, \mathcal{C}_n receives contribution from logarithms of m^2/Q^2 as well as constants. When $Q \gg m$, there arises the ambiguity of setting the renormalization scale in the α_s , whether to put it around m^2 or Q^2 may result in quite different answer for a fixed order calculation. Moreover, these large logarithms may potentially ruin the perturbative expansion in strong coupling constant. Solution to these problems lies beyond the scope of NRQCD. Needless to say, it is highly desirable that these large kinematic logarithms in the NRQCD matching coefficients can be identified and summed to all orders in α_s , in order to ameliorate perturbative expansion.

The cause for this symptom can be easily traced. The problem is that the hard quanta integrated out by NRQCD still contains two widely separated scales, accommodating the quantum fluctuations extending from the scale m to a much higher scale Q . It is natural to speculate, for such a multi-scale problem, whether one can divide this rather coarse region further into finer ones. To achieve this, one needs first identify various relevant degrees of freedom, then disentangle their contributions, and finally express \mathcal{C}_n in a factorized form.

The NRQCD factorization is justified mainly because b is heavy, $m \gg \Lambda_{\text{QCD}}$. As $Q \gg m$, the H moves almost with the speed of light in a natural reference frame such as the center-of-mass frame. Since the *heavy* b and \bar{b} move nearly along the light cone, they can be considered to be *light* in the kinematic sense. We thus are facing a challenge where the conflicting nature of “light” heavy quarks must be consistently incorporated.

From now on, we will specialize to the S -wave quarkonium production. For simplicity,

³ In fact, for some classes of exclusive charmonium production processes, it has been proved that NRQCD factorization may no longer hold beyond the leading order in m^2/Q^2 [14]. Therefore it seems not profitable, for both pragmatic and theoretical reason, to proceed beyond the leading power.

we will only consider the leading term in the NRQCD expansion, since \mathcal{O}_1 is the simplest NRQCD operator and constitutes the most important contribution. However, we would like to stress that the purpose of imposing this restriction is mainly for simplicity. Since our reasoning will be based on quite general ground, there should be no principal difficulty to proceed to higher orders in v expansion.

To compute the short-distance coefficient \mathcal{C}_1 , a convenient method is to consider the quark amplitude where H is replaced by a free $b\bar{b}$ pair sharing the same quantum numbers (color, spin, orbital angular momentum) as the leading Fock component of H . At the lowest-order in v , b and \bar{b} are forced to partition equally the total momentum of the pair, P . For definiteness P is supposed to move fast along the positive \hat{z} axis. Consider a generic loop diagram that contributes to the corresponding quark amplitude. The so-called “method of region” [15] can be utilized to help identify the relevant degrees of freedom in \mathcal{C}_1 . There is a “hard” region ⁴, in which the loop momentum scales as $p^\mu \sim Q$. In this region, the b quark can be treated as massless. There are several kinds of infrared modes, *soft* ($p^\mu \sim m$), *collinear* ($p^+ \sim Q, p^- \sim m^2/Q, p_\perp \sim m$), and *anti-collinear* ($p^+ \sim m^2/Q, p^- \sim Q, p_\perp \sim m$) ⁵. All these infrared modes have the virtuality of order m^2 , hence the mass of b quark must be retained in these infrared regions. The validity of NRQCD factorization guarantees there is no any overlap between these “infrared” quanta and those truly infrared modes intrinsic to NRQCD such as the potential mode, as can be readily distinguished by their virtualities.

It turns out that at the leading power of m^2/Q^2 , the soft and anti-collinear modes do not contribute to \mathcal{C}_1 [16]. Therefore the relevant dynamic degrees of freedom are only the hard and collinear modes, which resembles the hard exclusive processes involving a single hadron such as $\pi \rightarrow \gamma$ transition form factor [1]. Quite naturally, one can follow the standard collinear factorization theorem at leading twist to cast \mathcal{C}_1 as

$$\mathcal{C}_1 \left(\frac{m^2}{Q^2} \right) \sim T(x, Q, \mu) \otimes \hat{\phi}(x, m, \mu) + O(m^2/Q^2), \quad (2)$$

where T refers to the hard-scattering part involving massless quarks, and $\hat{\phi}$ can be viewed as the LCDA of a free $b\bar{b}$ pair with the same quantum numbers as H (Later we will give

⁴ We caution that the term *hard* here only refers to a subset of hard region in the sense of NRQCD, and the actual meaning of this term may vary depending on different context. Similarly the term “soft” may also bear different meaning in different places.

⁵ Our convention in defining the light-cone variable is $A^\pm = \frac{A^0 \pm A^3}{\sqrt{2}}$, and a four-vector A^μ is decomposed into $(A^+, A^-, \mathbf{A}_\perp)$ in this convention.

a rigorous definition for this LCDA). x is the fraction of plus-momentum carried by the b quark with respect to that of the pair, and the operation \otimes implies the convolution over x between two parts. μ is an arbitrary scale separating these two regions. It is important to note that the scales Q and m now become fully disentangled, *i.e.* the T only depends on Q but not on m , likewise the $\hat{\phi}$ cannot have any direct sensitivity on the hard scale Q .

It is worth emphasizing that $\hat{\phi}$ introduced here (to be concrete, at this moment let us specialize to the twist-2 LCDA of the $b\bar{b}(^1S_0^{(1)}, P)$ state), bears some important difference with the twist-2 LCDA of a pion. The latter is a genuinely nonperturbative object that is sensitive to the quantum fluctuation at hadronic scale, so that it can only be extracted from experiments or from some nonperturbative tools such as lattice simulation or QCD sum rules. On the contrary, m acts as the infrared cutoff in the former case. Since m is much greater than Λ_{QCD} , $\hat{\phi}$ thus can be reliably computed in perturbation theory. However, we note that, since both LCDAs of the $b\bar{b}(^1S_0^{(1)}, P)$ state and a pion possess the same ultraviolet behavior, they obey the same renormalization group (RG) equation.

The factorization formula (2) is expected to hold to all orders in α_s . Both of the two ingredients, T and $\hat{\phi}$ are perturbatively improvable. When $Q \gg m$, this light-cone-based method provides an alternative and efficient means to reproduce the NRQCD short-distance coefficient, including the logarithms as well as the constants, to any desired order in α_s .

At tree level, the hard part $T(x, Q)$ can be easily evaluated, and the $\hat{\phi}$ defined at the scale $\mu \sim m$ is simply $\delta(x - \frac{1}{2})$, to be compatible with the NRQCD matching condition. Using the standard evolution equation to evolve this LCDA into higher scale $\mu \sim Q$, combined with the tree-level result of T , Eq. (2) then automatically sums the leading logarithms to all orders in α_s .

In the remainder of the paper, we will test our understanding through two explicit examples of exclusive production of a bottomonium plus a photon. We will illustrate how the leading kinematic logarithms in the NRQCD short-distance coefficient can be identified and resummed within the framework of the factorization formula (2).

Example 1— $e^+e^- \rightarrow \eta_b\gamma$. Let us first consider the exclusive $\eta_b + \gamma$ production at high-energy electron-positron collision, say, at LEP experiment. Similar processes at B factory environment have been studied in a light-front model [17] and the NRQCD approach [18]. To our purpose, it suffices to focus on the decay of a highly virtual photon, $\gamma^*(Q) \rightarrow \eta_b(P) + \gamma(k)$. According to the NRQCD factorization, one can express the decay amplitude at the lowest

order in v as

$$\mathcal{M}[\gamma^* \rightarrow \eta_b + \gamma] = \widehat{\mathcal{M}}[\gamma^* \rightarrow b\bar{b}(^1S_0^{(1)}, P) + \gamma] \frac{\langle \eta_b | \psi^\dagger \chi | 0 \rangle}{\sqrt{2N_c m}}, \quad (3)$$

where ψ^\dagger and χ are two-component Pauli spinor fields. The factor associated with the NRQCD matrix element is chosen for convenience, mainly to compensate the fact that the particle states in the amplitude is relativistically normalized, but the η_b state in NRQCD matrix element conventionally assumes non-relativistic normalization [3]. It is worth mentioning that, the factor $\langle \eta_b | \psi^\dagger \chi | 0 \rangle / \sqrt{m}$ coincides with f_{η_b} , the η_b decay constant, to the lowest-order accuracy in α_s and v .

$\widehat{\mathcal{M}}$, being the short-distance coefficient, represents the quark amplitude when η_b is replaced by the free b and \bar{b} quarks that equally share the total momentum P , and have the spin-color wave function $\frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} \times \frac{\delta^{ij}}{\sqrt{N_c}}$, where $N_c = 3$ is the number of color and i, j represent color indices. The calculation is usually expedited by utilizing the covariant projection method [19], though alternative methods also exist [20, 21]. The result is

$$\widehat{\mathcal{M}}[\gamma^* \rightarrow b\bar{b}(^1S_0^{(1)}, P) + \gamma] = \sqrt{2N_c} \frac{e_b^2 e_b^2}{Q^2} \epsilon_{\mu\nu\alpha\beta} \varepsilon_{\gamma^*}^\mu \varepsilon_\gamma^{*\nu} Q^\alpha k^\beta F\left(\frac{m^2}{Q^2}\right), \quad (4)$$

where $e_b = -\frac{1}{3}$ is the fractional electric charge of b quark. We introduce F as a dimensionless form factor. It can be simultaneously expanded in powers of α_s and m^2/Q^2 :

$$F\left(\frac{m^2}{Q^2}\right) = \sum_{n=0}^{\infty} \sum_{l=0}^n C_{nl} \left(\frac{\alpha_s}{\pi}\right)^n \ln^l\left(\frac{Q^2}{m^2}\right) + O\left(\frac{m^2}{Q^2}\right), \quad (5)$$

where C_{nl} are constants. As explained before, the “higher twist” contributions suppressed by $1/Q^2$ are omitted. For this simple process, the leading logarithm at the n th loop order is of the form $\alpha_s^n \ln^n(Q^2/m^2)$, and our goal is to deduce C_{nn} analytically as well as resum their effects to all orders in α_s ⁶.

The difficulty of NRQCD matching computation rapidly grows once beyond the tree level, and the resulting analytic expressions of F are generally rather cumbersome, often plagued with special functions depending on m^2/Q^2 in a complicated way (for some concrete examples on single and double charmonium production to one-loop order, see Ref. [22, 23,

⁶ Note that the function F is complex-valued in this process. Its imaginary part can contain sub-leading logarithms. However, insofar as the leading logarithms are concerned, we are allowed to ignore the contributions from the imaginary part.

24]). However, the expanded form as indicated in (5) is always very simple. It is desirable if F can be reproduced in a more efficient way. As discussed earlier, this is indeed possible, if we try to disentangle “hard” and “collinear” quanta in the NRQCD matching coefficients. For this example, a factorization formula in complete analogy to the $\pi-\gamma$ form factor is expected to hold:

$$F\left(\frac{m^2}{Q^2}\right) = \int_0^1 dx T(x, Q, \mu) \hat{\phi}(x, m, \mu) + O\left(\frac{m^2}{Q^2}\right), \quad (6)$$

where μ is an arbitrary factorization scale that separates the hard and collinear contributions. The hard function T is identical to what appears in the $\pi-\gamma$ form factor, which can be expanded as $T = T^{(0)} + \frac{\alpha_s}{\pi} T^{(1)} + \dots$. The tree-level result, $T^{(0)}(x, Q) = \frac{1}{x} + \frac{1}{1-x}$, is well known, and $T^{(1)}$ has also been available for a long time [25, 26]. The “long-distance” contribution is characterized by $\hat{\phi}$, the twist-2 LCDA of a free $b\bar{b}$ pair in the spin-color state $^1S_0^{(1)}$ and with total momentum P halved by its two constituents. This LCDA admits a definition in term of operator matrix element [2]:

$$\hat{\phi}(x, m, \mu) = -\frac{1}{\sqrt{2N_c}} \int \frac{dw^-}{2\pi} e^{-ixP^+w^-} \langle b\bar{b}(^1S_0^{(1)}, P) | \bar{b}(0, w^-, \mathbf{0}_\perp) \gamma^+ \gamma_5 b(0) | 0 \rangle, \quad (7)$$

where for simplicity we have suppressed the light-like gauge link. This definition is boost invariant along the \hat{z} axis. The $\hat{\phi}$ can be systematically expanded perturbatively, $\hat{\phi} = \hat{\phi}^{(0)}(x, m, \mu) + \frac{\alpha_s}{\pi} \hat{\phi}^{(1)}(x, m, \mu) + \dots$, here μ can be identified with the renormalization scale of the operator matrix element. Its tree-level result is very simple, $\hat{\phi}^{(0)}(x, \mu \sim m) \equiv \hat{\phi}^{(0)}(x) = \delta(x - \frac{1}{2})$, embodying the NRQCD matching condition. For the investigations on $\hat{\phi}^{(1)}$, one may consult Ref. [10, 11]. Substituting the tree-level expressions of $T^{(0)}$ and $\hat{\phi}^{(0)}$ into (6), one then obtains $C_{00} = 4$, which agrees with the result derived from the standard NRQCD matching [18].

The invariance of physical amplitude about the choice of the factorization scale μ leads to the RG equation, which is conventionally referred to as Brodsky-Lepage (BL) equation [27]. This equation governs the evolution of the LCDA from one scale to another one, through which the collinear logarithms can be resummed. The BL equation reads:

$$\mu^2 \frac{\partial}{\partial \mu^2} \hat{\phi}(x, \mu) = \frac{\alpha_s(\mu^2)}{\pi} \int_0^1 dy \frac{C_F}{2} V_0(x, y) \hat{\phi}(y, \mu), \quad (8)$$

where

$$V_0(x, y) = \left[\frac{1-x}{1-y} \left(1 + \frac{1}{x-y} \right) \theta(x-y) + \frac{x}{y} \left(1 + \frac{1}{y-x} \right) \theta(y-x) \right]_+, \quad (9)$$

is the kernel governing the evolution of LCDA of a helicity-zero meson. $C_F = \frac{N_c^2 - 1}{2N_c}$ is the Casimir for $SU(N_c)$ fundamental representation, the subscript “+” implies the familiar “plus” prescription. This kernel can be interpreted as the amplitude for a quark with fractional plus-momentum x and antiquark with $1 - x$ to become a quark with fractional plus-momentum y and antiquark with $1 - y$ by the exchange of one collinear gluon.

As is well known, the kernel V_0 admits the eigenfunctions $G_n(x) \equiv x(1-x)C_n^{(3/2)}(2x-1)$, which are Gegenbauer polynomials of order $\frac{3}{2}$ multiplied by the weight function $x(1-x)$:

$$\int_0^1 dy V_0(x, y) G_n(y) = -\gamma_n G_n(x), \quad (10a)$$

$$\gamma_n = \frac{1}{2} + 2 \sum_{j=2}^{n+1} \frac{1}{j} - \frac{1}{(n+1)(n+2)}. \quad (10b)$$

It is a standard procedure to decompose the $\hat{\phi}(x)$ in the basis of G_n :

$$\hat{\phi}(x, \mu) = \sum_{n=0}^{\infty} \hat{\phi}_n(\mu) G_n(x), \quad (11a)$$

$$\hat{\phi}_n(\mu) = \frac{4(2n+3)}{(n+1)(n+2)} \int_0^1 dx C_n^{(3/2)}(2x-1) \hat{\phi}(x, \mu). \quad (11b)$$

At leading logarithmic accuracy, one can express the form factor as the infinite sum of Gegenbauer moments:

$$F\left(\frac{m^2}{Q^2}\right)_{\text{LL}} = \int_0^1 dx T^{(0)}(x) \hat{\phi}^{(0)}(x, Q^2) = \sum_{n=0}^{\infty} \hat{\phi}_{2n}^{(0)}(Q^2). \quad (12)$$

All the odd Gegenbauer moments vanish. In deriving (12), we have made use of the fact $\int_0^1 dx C_{2n}^{(3/2)}(2x-1) = 1$ for any integer n . Using $C_{2n}^{(3/2)}(0) = \frac{(-1)^n (2n+1)!!}{(2n)!!}$, and evolving the moments from the low scale $\mu \sim m$ to Q , we obtain

$$F\left(\frac{m^2}{Q^2}\right)_{\text{LL}} = \sum_{n=0}^{\infty} \hat{\phi}_{2n}^{(0)} \left(\frac{\alpha_s(Q^2)}{\alpha_s(m^2)} \right)^{d_{2n}}, \quad (13a)$$

$$\hat{\phi}_{2n}^{(0)} = 4(-1)^n (4n+3) \frac{(2n-1)!!}{(2n+2)!!}. \quad (13b)$$

The anomalous dimension for each moment is $d_n = 2 C_F \gamma_n / \beta_0$, where γ_n is given in (10b). $\beta_0 = \frac{11}{3}N_c - \frac{2}{3}n_f$ is the one-loop coefficient of QCD β function and n_f is number of active quark flavors. The running strong coupling constant is given by $\alpha_s(Q^2) = \frac{4\pi}{\beta_0 \ln(Q^2/\Lambda_{QCD}^2)}$. $\hat{\phi}_{2n}^{(0)}$ is an alternating series and its magnitude declines very slowly ($\propto 1/\sqrt{n}$ asymptotically [11]),

and the convergence of the sum becomes extremely slow. Practically, one has to include an extraordinary number of terms in the sum to warrant the convergence of the result.

In Fig. 1 we show the effect of leading logarithm resummation, (13a), for a wide range of values of Q . One can clearly see that this RG-improved result seems to only have a minor numerical impact, perhaps less important than the constant terms in the fixed order calculation. As $Q^2 \rightarrow \infty$, only the first moment in (13a) can survive since $d_0 = 0$ and all the other d_{2n} are positive, therefore $F \rightarrow \frac{3}{2}C_{00}$. However, as might be inferred from Fig. 1, to reach this asymptotic result, an unphysically large Q (*i.e.* $\gg M_{\text{Planck}}$) is required.

Even though leading logarithms have been summed to all orders in α_s , it would still be illuminating if the coefficient of each leading logarithm at n -th loop order, C_{nn} , can be deduced. To this end, it is more convenient to work in the x space than the moment space. We can rewrite F as:

$$F\left(\frac{m^2}{Q^2}\right)_{\text{LL}} = \int_0^1 dx T^{(0)}(x) \hat{\phi}^{(0)}(x, Q) = \int_0^1 dx T^{(0)}(x) \exp[\kappa C_F V_0 \star] \hat{\phi}^{(0)}(x), \quad (14)$$

where

$$\kappa \equiv \frac{2}{\beta_0} \ln\left(\frac{\alpha_s(m^2)}{\alpha_s(Q^2)}\right) = \frac{\alpha_s(Q^2)}{2\pi} \ln\left(\frac{Q^2}{m^2}\right) + \beta_0 \frac{\alpha_s^2(Q^2)}{(4\pi)^2} \ln^2\left(\frac{Q^2}{m^2}\right) + \dots \quad (15)$$

In the second equation of Eq. (14), we have substituted the symbolic solution of the BL equation for $\hat{\phi}(x, Q)$, which is evolved from the infrared scale at $\mu \sim m$ to the high scale $\mu \sim Q$. The \star operation implies that, upon the expansion of the exponential, the resulting products of the argument of the exponential are to be treated as convolutions. It is worth noting that, Eq. (14) is essentially the statement that the moment-space amplitude exponentiates, as indicated in (13a).

We can expand equation (14) iteratively:

$$\begin{aligned} F\left(\frac{m^2}{Q^2}\right)_{\text{LL}} &= \int_0^1 dx T^{(0)}(x) \hat{\phi}^{(0)}(x) + \kappa C_F \int_0^1 dx \int_0^1 dy T^{(0)}(x) V_0(x, y) \hat{\phi}^{(0)}(y) \\ &\quad + \frac{\kappa^2 C_F^2}{2!} \int_0^1 dx \int_0^1 dy \int_0^1 dz T^{(0)}(x) V_0(x, y) V_0(y, z) \hat{\phi}^{(0)}(z) + \dots \end{aligned} \quad (16)$$

Equation (16) admits a clear physical picture how the leading logarithms are built up. At the hard vertex, b and \bar{b} are produced to carry arbitrary fractional plus-momentum x and $1 - x$, and transverse momenta of order Q . In the course of evolution, they keep exchanging

collinear gluons ⁷ and reshuffle the respective fractional plus-momenta, and gradually decrease the transverse momenta. When ready to transition to a physical η_b state, they finally adjust themselves to become nearly collinear and halve the total momentum.

We choose the order of the multiple integration in (16) from the left to the right, and leaves the integration over the $\hat{\phi}^{(0)}$ in the last step. With the help of (A1a) and (A1b), we are able to deduce the leading logarithms through the two-loop order:

$$F\left(\frac{m^2}{Q^2}\right)_{\text{LL}} = C_{00} \left\{ 1 + \frac{C_F \alpha_s(Q^2)}{4\pi} \ln\left(\frac{Q^2}{m^2}\right) (3 - 2 \ln 2) \right. \\ \left. + C_F \frac{\alpha_s^2(Q^2)}{(4\pi)^2} \ln^2\left(\frac{Q^2}{m^2}\right) \left[\beta_0 \left(\frac{3}{2} - \ln 2 \right) + C_F \left(\frac{9}{2} - \frac{\pi^2}{6} - 8 \ln 2 + \ln^2 2 \right) \right] + \dots \right\}, \quad (17)$$

where the coefficient C_{11} agrees with the asymptotic expansion of the exact NLO correction to the NRQCD short-distance coefficient [28], and can also be read off by imposing $x = \frac{1}{2}$ in $T^{(1)}(x)$ in Ref. [26]. Our prediction to C_{22} is new. From Fig. 1, one can see that, for a wide range of Q/m , the leading logarithm approximation through two-loop order is already quite close to the resummed results, (13a).

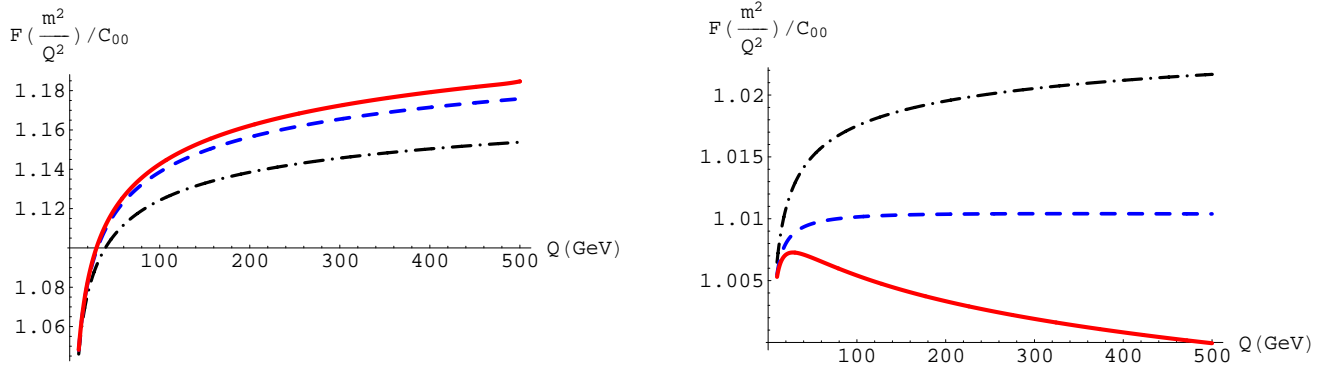


FIG. 1: The form factor $F(m^2/Q^2)$ as a function of Q for fixed $m_b = 4.7$ GeV. The left panel is for $\gamma^* \rightarrow \eta_b + \gamma$, and the right panel for $h \rightarrow \Upsilon + \gamma$. We have fixed $n_f=5$, $\beta_0 = \frac{23}{3}$ and $\Lambda_{\text{QCD}} = 100$ MeV. Various curves represent the improved results to different extent in leading logarithm approximation. The dot-dashed line only includes the logarithm to one-loop order, the dashed line includes the leading logarithms through two-loop order, and the solid line sums the leading logarithms to all orders in α_s .

⁷ We have tacitly made an important simplification. The kernel V_0 we use is valid only for the flavor nonsinglet where annihilation of $q\bar{q}$ is not allowed. In our case, it is possible that the quark pair in the intermediate loop to annihilate into two gluons, then recreates $b\bar{b}$ pair. These types of singlet diagrams first occur at two loop. In this work, we neglect the logarithms arising from such singlet contributions.

Example 2—Higgs radiative decay into Υ . It is tempting to test our understanding in a slightly different situation, *e.g.* a process that a transversely polarized Υ is produced at leading power. Let us consider $h(Q) \rightarrow \Upsilon(P) + \gamma(k)$. In view of its clean signature, this decay channel may be potentially useful to search for the Higgs boson in the forthcoming CERN Large Hadron Collider (LHC) experiment, were it not completely swallowed by the much more copious background. In accordance with the NRQCD factorization, we can express the amplitude at the lowest order in v as ⁸

$$\mathcal{M}[h \rightarrow \Upsilon + \gamma] = \widehat{\mathcal{M}}[h \rightarrow b\bar{b}(^3S_1^{(1)}, P) + \gamma] \frac{\langle \Upsilon(\epsilon) | \psi^\dagger \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon} \chi | 0 \rangle}{\sqrt{2N_c m}}. \quad (18)$$

It might be worth pointing out that, the NRQCD matrix element $\langle \Upsilon(\epsilon) | \psi^\dagger \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon} \chi | 0 \rangle / \sqrt{m}$ coincides with f_Υ , Υ decay constant, to the lowest-order accuracy in α_s and v .

Analogous to the first example, $\widehat{\mathcal{M}}$ here represents the parton amplitude when Υ is replaced by the free b and \bar{b} quarks in the $^3S_1^{(1)}$ state, each of which carries half of the total momentum P . It can be expressed as follows:

$$\widehat{\mathcal{M}}[h \rightarrow b\bar{b}(^3S_1^{(1)}, P) + \gamma] = -\sqrt{2N_c} \frac{e e_b}{2} g_{hb\bar{b}} \varepsilon_\Upsilon^* \cdot \varepsilon_\gamma^* F\left(\frac{m^2}{Q^2}\right), \quad (19)$$

where $Q^2 = M_h^2$, $g_{hb\bar{b}} = m/v$ is the Yukawa coupling between Higgs and b quarks, and $v = 246$ GeV implies the vacuum expectation value of the Higgs field. Since the photon must be transversely polarized, so is the Υ .

The dimensionless form factor F here is expected to also satisfy a collinear factorization formula similar to (6). The tree-level hard coefficient function $T^{(0)}$ is the same as the preceding example, but here one should use the twist-2 LCDA of the transversely polarized $b\bar{b}(^3S_1^{(1)}, P)$ state, which can be defined as the following operator matrix element [2]:

$$\begin{aligned} \hat{\phi}_\perp(x, m, \mu) &= \frac{i}{\sqrt{2N_c}} \int \frac{dw^-}{2\pi} e^{-ixP^+w^-} \\ &\times \langle b\bar{b}(^3S_1^{(1)}, P, \lambda_\perp) | \bar{b}(0, w^-, \mathbf{0}_\perp) \sigma^{+\mu} \epsilon_\mu(\lambda_\perp) b(0) | 0 \rangle. \end{aligned} \quad (20)$$

At the lowest order in α_s , $\hat{\phi}_\perp^{(0)}(x, \mu \sim m) \equiv \hat{\phi}^{(0)}(x) = \delta(x - 1/2)$. Thus in this example, we also get $C_{00} = 4$. The $\hat{\phi}_\perp$ also satisfies the BL equation (8), except the kernel V_0 should be

⁸ Our main concern here is for the illustrative purpose, so we do not target at a serious phenomenological analysis of this decay channel. For instance, we have not included the fragmentation contribution from $h \rightarrow \gamma^* \gamma \rightarrow \Upsilon \gamma$.

replaced by a different one:

$$V_{\perp}(x, y) = V_0(x, y) - \left[\frac{1-x}{1-y} \theta(x-y) + \frac{x}{y} \theta(y-x) \right]. \quad (21)$$

This new kernel permits the same eigenfunctions $G_n(x)$ as given in Eq. (10a), but the corresponding eigenvalues are slightly different from Eq. (10b), *i.e.* $\tilde{\gamma}_n = \frac{1}{2} + 2 \sum_{j=2}^{n+1} \frac{1}{j}$.

Completely analogous to the preceding example, we can express the form factor F as the infinite sum of Gegenbauer moments, with evolution effect taken into account:

$$\begin{aligned} F \left(\frac{m^2}{Q^2} \right)_{\text{LL}} &= \int_0^1 dx T^{(0)}(x) \hat{\phi}_{\perp}^{(0)}(x, Q^2) \\ &= \sum_{n=0}^{\infty} \hat{\phi}_{\perp 2n}(Q^2) = \sum_{n=0}^{\infty} \hat{\phi}_{2n}^{(0)} \left(\frac{\alpha_s(Q^2)}{\alpha_s(m^2)} \right)^{\tilde{d}_{2n}}, \end{aligned} \quad (22)$$

where the anomalous dimension for the moment is $\tilde{d}_n = 2 C_F \tilde{\gamma}_n / \beta_0$.

We also show the effect of leading logarithm resummation, (22), for a wide range of values of Q in Fig. 1. One can clearly see that this RG-improved result seems to have an even minor impact compared with the preceding example ⁹. Since all the \tilde{d}_{2n} are positive, as $Q^2 \rightarrow \infty$, one expects $F \rightarrow 0$. However, as indicated in Fig. 1, to reach this asymptotic result, an unphysically large Higgs mass is required.

We can also iteratively derive the closed form for the leading logarithms occurring at the n -th loop order, by simply replacing the kernel V_0 in (16) by V_{\perp} . Using (A3a) and (A3b), we can predict the leading logarithms through the two-loop order:

$$\begin{aligned} F \left(\frac{m^2}{Q^2} \right)_{\text{LL}} &= C_{00} \left\{ 1 + \frac{C_F \alpha_s(Q^2)}{4\pi} \ln \left(\frac{Q^2}{m^2} \right) (3 - 4 \ln 2) \right. \\ &\quad \left. + C_F \frac{\alpha_s^2(Q^2)}{(4\pi)^2} \ln^2 \left(\frac{Q^2}{m^2} \right) \left[\beta_0 \left(\frac{3}{2} - 2 \ln 2 \right) + C_F \left(\frac{9}{2} - 12 \ln 2 + 4 \ln^2 2 \right) \right] + \dots \right\}. \end{aligned} \quad (23)$$

As a check, we have explicitly computed the logarithmical contribution in $T^{(1)}(\frac{1}{2})$ for this process ¹⁰. It agrees with C_{11} given above. As one can tell from Fig. 1, even though including the leading logarithm at two-loop order noticeably modifies the one-loop result, the overall effect of logarithms is too modest to be phenomenologically relevant.

⁹ Our investigations on these two examples show that, even when $Q \gg m$, leading logarithm resummation may not play a significant role. This finding provides a counterexample to the assertion made in [8].

¹⁰ There is a subtlety arising in this process. Since the composite operator $\bar{b}b$ acquires an anomalous dimension, the renormalization procedure has to be performed to the $T^{(1)}$, where a renormalization scale μ_R will enter and give rise to the logarithm like $\ln(Q^2/\mu_R^2)$. This type of logarithm has a very different origin from the collinear logarithm we are interested, so will be eliminated by choosing $\mu_R \sim Q$.

Summary. In this paper we suggest that, for a class of exclusive single-quarkonium production processes, the short-distance coefficient in the leading order NRQCD expansion can be further separated into “hard” and “collinear” degrees of freedom. As a consequence, the corresponding NRQCD matching coefficient can be expressed as the convolution between a hard coefficient function and a perturbatively calculable LCDA of a properly-chosen free quark-antiquark pair. The procedure of refactorization has both conceptual and technical advantages over the conventional NRQCD matching calculation. For instance, we have shown that large kinematic logarithms can be readily summed to all orders in α_s within this scheme, which is otherwise difficult to accomplish. This strategy also provides an efficient means to reproduce the NRQCD matching coefficients when proceeding beyond the tree level (for an explicit one-loop illustration, see [16]).

We believe that the idea of refactorization is rather general and deserves to be further explored in other more complicated cases. For example, the NRQCD short-distance coefficients associated with exclusive P -wave quarkonium production and relativistic corrections to S -wave quarkonium production, should also be amenable to a similar collinear factorization formula. It is conceivable that exclusive double charmonium production processes like $e^+e^- \rightarrow J/\psi + \eta_c$, may even be approachable from this angle. However, since such processes often violate the helicity selection rule and the amplitude usually starts at subleading power of m^2/Q^2 , consequently higher-twist LCDAs must be introduced [6, 7, 8]. This may pose some great challenge to fulfill factorization and evolution.

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Note added. After the manuscript was submitted, we became aware of a work done by Shifman and Vysotsky in 1981 [29], who had considered two basically identical processes as ours, and also attempted to resum the respective leading kinematic logarithms. Though differing in technical taste, their and our approaches are essentially the same, both derived from the light-cone OPE. Nevertheless, our approach seems easier to follow in practice. For the first process $e^+e^- \rightarrow \eta_b\gamma$, they also considered the mixing effects due to the two-gluon

Fock components of η_b , which we have not. Both of works agree on the one-loop coefficient of leading logarithms in this process. Our approach enables one to readily deduce the two-loop coefficient as well, but it would be a laborious undertaking for their formalism to achieve this. For the second process $h \rightarrow \Upsilon\gamma$, they combined the collinear logarithms with those logarithms arising from the running quark mass. After subtracting the latter type of logarithms, their results agree with ours. It should be noted, however, the strength of refactorization proposed in this work is not only limited to summing large logarithms. As has been stressed throughout the paper, our approach will serve an efficient and systematic tool to reproduce the NRQCD short-distance coefficients to any desired order in α_s , including the logarithms as well as the constants, in various exclusive quarkonium production processes. In this regard, we feel that our work has gone one step farther than Ref. [29].

APPENDIX A: USEFUL MATHEMATICAL FORMULAS

In *Example 1*, in order to iteratively deduce the leading logarithms through the two-loop order from (16), we need know the following integrals:

$$\int_0^1 dx T^{(0)}(x) V_0(x, y) = \frac{1}{y(1-y)} \left[\frac{3}{2} + (1-y) \ln y + y \ln(1-y) \right], \quad (\text{A1a})$$

$$\begin{aligned} \int_0^1 dx \int_0^1 dy T^{(0)}(x) V_0(x, y) V_0(y, z) = & \frac{1}{z(1-z)} \left[\frac{9}{4} - \frac{\pi^2}{6} + \ln z + \ln(1-z) + 2(1-z) \ln z \right. \\ & \left. + 2z \ln(1-z) + (1-z) \ln^2 z + z \ln^2(1-z) + z \text{Li}_2(z) + (1-z) \text{Li}_2(1-z) \right], \end{aligned} \quad (\text{A1b})$$

where Li_2 denotes dilogarithm. Eq. (A1a) can be found in Ref. [26], and (A1b) is new.

Expand the Gegenbauer-moment-summation formula (13a) in d_{2n} , truncate the power series in $\alpha_s(Q^2) \ln(Q^2/m^2)$ to the second order, and match them onto Eq. (17). Enforcing the mutual equivalence leads to the following mathematical identities:

$$\sum_{n=0}^{\infty} \hat{\phi}_{2n}^{(0)} = 4. \quad (\text{A2a})$$

$$\sum_{n=0}^{\infty} \gamma_{2n} \hat{\phi}_{2n}^{(0)} = -2(3 - 2 \ln 2). \quad (\text{A2b})$$

$$\sum_{n=0}^{\infty} \gamma_{2n}^2 \hat{\phi}_{2n}^{(0)} = 9 - \frac{\pi^2}{3} - 16 \ln 2 + 2 \ln^2 2. \quad (\text{A2c})$$

Analogously, in *Example 2*, to iteratively acquire the leading logarithms up to two-loop

order, we need to know the following integrals:

$$\int_0^1 dx T^{(0)}(x) V_{\perp}(x, y) = \frac{1}{y(1-y)} \left[\frac{3}{2} + \ln y + \ln(1-y) \right], \quad (\text{A3a})$$

$$\begin{aligned} & \int_0^1 dx \int_0^1 dy T^{(0)}(x) V_{\perp}(x, y) V_{\perp}(y, z) \\ &= \frac{1}{z(1-z)} \left[\frac{9}{4} + 3 \ln z + 3 \ln(1-z) + \ln^2 z + \ln^2(1-z) \right]. \end{aligned} \quad (\text{A3b})$$

Expand the Gegenbauer-moment-summation formula (22) in \tilde{d}_{2n} , truncate the power series in $\alpha_s(Q^2) \ln(Q^2/m^2)$ to the second order, and compare them with (23). The mutual equivalence demands that the following identities must hold:

$$\sum_{n=0}^{\infty} \tilde{\gamma}_{2n} \hat{\phi}_{2n}^{(0)} = -2(3 - 4 \ln 2), \quad (\text{A4a})$$

$$\sum_{n=0}^{\infty} \tilde{\gamma}_{2n}^2 \hat{\phi}_{2n}^{(0)} = 9 - 24 \ln 2 + 8 \ln^2 2. \quad (\text{A4b})$$

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